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NATIONAL BUREAU OF STANDARDS REPORT

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ON ABSOLUTE CONVERGENCE OF LINEAR ITERATION PROCESSES

by

A. M. Ostrowski

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ON ABSOLUTE CONVERGENCE OF LINEAR ITERATION PROCESSES I

By A. M. Ostrowski

1. In solving the linear system $A\xi' = \eta'$ by iteration (where $A = (a_{\mu\nu})$ is a quadratic matrix of order n as are all matrices considered later, save, of course, vectors) we establish a sequence of vectors ξ_k, approximating vectors to the solution vector ξ; the corresponding <u>residual vectors</u> ρ_k are given by $A\xi_k' - \eta' = \rho_k'$. We assume in what follows that the diagonal elements $a_{\mu\mu}$ are $= 1$.

In the <u>single step iteration</u> at each step only <u>one</u> component of ξ_k is changed; if the components of ξ_k are $x_\nu^{(k)}$, then to each k corresponds a leading index N_k such that

$$(1)\quad x_\nu^{(k+1)} = x_\nu^{(k)} \quad (\nu \neq N_k), \quad x_{N_k}^{(k+1)} - x_{N_k}^{(k)} = \delta_k.$$

Usually δ_k is so chosen that the corresponding components of ρ_{k+1} becomes 0; if the components of ρ_k are $r_\nu^{(k)}$, we have then $\delta_k = -r_{N_k}^{(k)}$.

In the case the so-called <u>incomplete relaxation</u> is allowed we take

$$(2)\quad \delta_k = q_k r_{N_k}^{(k)}, \quad (0 < q_k < 2);$$

for $q_k < 1$ we have <u>under relaxation</u> and for $q_k > 1$ we have <u>over relaxation.</u>

2. A generalization of the single step iteration is the <u>group iteration</u>; here at each step the indices 1, ..., n are decomposed into two groups, <u>active indices</u> (α or γ) and <u>passive indices</u> (β) and in passing from ξ_k to ξ_{k+1} the components with passive indices are not changed, while for the change of the components with active indices the following rule is used. Define for the active indices γ the constants $\delta_\gamma^{(k)}$ from the set of equations

$$(3) \quad \sum_\gamma a_{\alpha\gamma}\,\delta_\gamma^{(k)} = -\,r_\alpha^{(k)}$$

where α and γ run over all active indices, and take

$$(4) \quad x_\beta^{(k+1)} = x_\beta^{(k)}, \quad x_\alpha^{(k+1)} - x_\alpha^{(k)} = q_\alpha^{(k)}\,\delta_\alpha^{(k)}, \; (0 < q_\alpha^{(k)} < 2).$$

An essentially different type of linear iteration is the <u>Jacobian</u> iteration described by the formulae ($\eta = (Y_1, \ldots, Y_n)$)

$$(4a) \quad x_\mu^{(k+1)} = Y_\mu - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^{n} a_{\mu\nu}\, x_\nu^{k}.$$

(See Argelander [1] and Jacobi [4]).

3. While in the iteration (4) no freedom is left save the choice of the starting vector ξ_1, in using the single step of the group iteration the choice of the leading index N_k or more generally of the active indices at each step - we call this the <u>steering of the process</u> - becomes the essential problem. If this choice depends on the values of the components ρ_k we call it the <u>relaxation processes</u> (introduced by Gauss [2], see also references to Gauss in Gerling [3] and later discussed

by Seidel [17] and rediscovered by Southwell [18]).[1] If the sequence of the leading indices or more generally of the groups of active indices is periodic, then, if no index is missing (while some of them may occur several times in a period) we speak of the <u>periodic steering</u>; if in particular, in a complete period each index occurs exactly once we speak of the <u>cyclic steering.</u> The cyclic steering appears to have been considered for the first time by Nekrassoff [8] although it was mentioned by Seidel [17] who did not recommend it.[2] The cyclic group iteration was first considered by Pizzetti [15] while the group relaxation was recommended by Gauss (see the reference in Gerling [3]). Finally we speak of the <u>free steering</u> if the sequence of the leading indices and active indices is chosen in an arbitrary way provided each index occurs an infinite number of times.[3] The incomplete relaxation is used in the practical computing but it has not been as yet theoretically discussed.

4. These iteration schemes arose from the method of the least squares. For a symmetric definite matrix the convergence of the single step iteration in the case of the steering by relaxation was first proved by Seidel [17]. Pizzetti [15] proved the convergence of the cyclic single step and group iteration, while Reich [16] showed that

for a symmetric matrix the cyclic single step iteration always diverges if the corresponding form is not a definite one.

5. In the case of non-symmetric matrices the first sufficient conditions for the convergence of the cyclic one "step iteration" were given in 1892:

(5) $\sum_{\mu}' |a_{\mu\nu}| < 1$ (Mehmke)[5], $\sum_{\nu}' |a_{\mu\nu}| < 1$ (Nekrassoff)[6].

Other sufficient conditions were given by Nekrassoff [9] and later other writers (see the references in [20]). These conditions contain usually not the elements $a_{\mu\nu}$ themselves but only their moduli and it was then in any case noticed that the same conditions are also sufficient for the convergence of the Jacobian iteration (4a). In what follows we will say that an iteration process as described above for a fixed choice of the active indices is absolutely convergent if it is convergent for any choice of the starting vector ξ_1 and if it remains so when the elements $a_{\mu\nu}$ of the matrix are multiplied by arbitrary numbers of modulus one. The conditions (5) and analogous conditions are then obviously sufficient conditions for the absolute convergence of the cyclic one step iteration.

In this note we will characterize completely all matrices A for which the cyclic one step iteration is absolutely convergent.

6. To a general matrix $A = (a_{\mu\nu})$ corresponds in a unique way the matrix

(6) $A_\beta = (|a_{\mu\nu}|(2\delta_{\mu\nu}-1)) \qquad \delta_{\mu\nu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$,

the adjoint matrix to A [11]. The matrix A is called of H type or an H matrix if $|A_\beta| > 0$ while all coaxial minors of A_β of all orders are not negative.

Then we obtain as the necessary and sufficient condition for the absolute convergence of the single step cyclic iteration, that the <u>matrix</u> A <u>is an</u> H <u>matrix</u>. More precisely we have the two more general theorems:

I. If the cyclic one step iteration converges for the matrix A_β for any choice of starting value ξ_1 , A <u>is an H</u> matrix.

II. If A is an H matrix then the cyclic one step iteration and even the one step iteration and the group iteration with free steering are convergent for the matrix A for any choice of the starting vector. Further, the incomplete relaxation is allowed to the following extent: let t_1 and t_2 be positive numbers where $t_1 < 1$ and t_2 is sufficiently small; then the one step iteration remains convergent if the vectors q_k satisfy the condition

(7) $t_1 \leq q_k \leq 1 + t_2 \qquad k = (1,2,\ldots\ldots)$.

In the case of the group iteration the factors $q_k^{(\gamma)}$ must satisfy the condition

(8) $|q_k^{(\gamma)} - 1| \leqslant t_2$.

A corresponding result holds also for the Jacobian iteration (4a).

III. A necessary and sufficient condition for the absolute convergence of the Jacobian iteration for any choice of the starting value ξ_1 is that the matrix A is an H matrix.

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7. There are many sufficient conditions for the H type matrices which, without being necessary, are often very easy to apply. We use the following notations:

(9) $$Z_\mu = \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} |a_{\mu\nu}|, \qquad S_\mu = \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} |a_{\nu\mu}|$$

(10) $$Z_\mu^{(p)} = \Big[\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} |a_{\mu\nu}|^p\Big]^{1/p},$$

(11) $$m_\mu = \operatorname*{Max}_{\nu\neq\mu} |a_{\mu\nu}|, \quad m = \operatorname*{Max}_{\mu>\nu} \left|\frac{a_{\mu\nu}}{a_{\mu\mu}}\right|, \quad M = \operatorname*{Max}_{\mu<\nu} \left|\frac{a_{\mu\nu}}{a_{\mu\mu}}\right| .$$

Then each of the following conditions (12) - (16) is sufficient for the matrix A being of the H type:

(12) $$Z_\mu < |a_{\mu\mu}| \quad (\mu = 1, \ldots, n);$$

(13) $$Z_\mu^{\alpha} \; S_\mu^{1-\alpha} < |a_{\mu\mu}| \quad (\mu = 1, \ldots, n)$$

for a convenient α, $0 \leq \alpha \leq 1$ [12];

(14) $$\sum_{\mu=1}^{n} \frac{1}{1+\left(\frac{|a_{\mu\mu}|}{Z_\mu^{(p)}}\right)^q} < 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \quad [13]$$

(15) $$\sum_{\mu=1}^{n} \frac{m_\mu}{|a_{\mu\mu}| + m_\mu} < 1 ; \qquad [13]$$

(16) $$m < M, \quad \frac{m}{(1+m)^n} < \frac{M}{(1+M)^n}. \qquad [14]$$

In interchanging the rows with the columns we obtain a further set of sufficient conditions.[4)]

8. A single step or group iteration scheme with a periodic steering is equivalent to a matrix iteration of the type $\xi'_{K+1} = A_1 \xi'_K + A_2 \eta'$, if the result of all steps contained in a period is interpreted as the passage from a vector ξ_K to a vector ξ_{K+1}. In the case of the cyclic single step iteration such a representation can be obtained in writing the matrix A as the sum L + D + R of the matrix L containing zeros on the diagonal and to the right of it, the matrix D containing zeros off the diagonal and the matrix R containing zeros on the diagonal and to the left of it. Then the matrix interpretation of the cyclic single step iterations corresponds to the formula

$$\xi'_{K+1} = -(L+D)^{-1} R \xi'_K + (L+D)^{-1} \eta' \;;$$

and from that the necessary and sufficient condition for the convergence of this iteration can be obtained in the form that the maximum modulus of the roots of the equation $|\lambda L + \lambda D + R| = 0$ remains < 1 [8,9]. We will denote this maximum modulus as the Nekrassoff number of A.

9. On the other hand a corresponding condition has been derived for the convergence of the Jacobian iteration; in this case it is necessary and sufficient for the convergence that the maximum modulus of the roots of the equation $|\lambda D + L + R| = 0$ is < 1 [7].

We will consider in particular the maximum modulus of the roots of this equation formed for the adjoint matrix A_β of A; this number will be denoted by σ_A and called the <u>Jacobian constant</u> of the matrix A. Then we can write the necessary and sufficient condition for the absolute convergence of the Jacobian iteration of the matrix A in the form $\sigma_A < 1$. This is equivalent to the theorem that $\sigma_A < 1$ is necessary and sufficient for the matrix A to be of the H type.

On the other hand it can be proved that if $\sigma_A < 1$, the Nekrassoff number of A remains $\leq \sigma_A$.

10. These results can be generalized in the following way. We call a matrix S a <u>truncated part</u>[5)] of the matrix A if the elements of S are obtained in multiplying the corresponding elements of A by arbitrary factors from the interval $\langle 0, 1\rangle$.

Let now the matrix A be decomposed into the sum of two truncated parts U, V: $A = U + V$, and suppose that all diagonal elements of V vanish; let $A_\beta = U_\beta + V_\beta$ be the corresponding decomposition of the adjoint matrix A_β. Assume now that U_β is not singular and write $\tilde{H} = -U_\beta^{-1} V_\beta$; then for the convergence of the stationary linear iteration

$$\xi'_{k+1} = \tilde{H}\, \xi'_k + (E - \tilde{H})\, A^{-1} \eta'$$

for an arbitrary starting vector ξ_1 it is <u>necessary</u> that A is an H matrix.

On the other hand, if A is an H matrix and if we consider for each k a particular decomposition of the above type $A = U_k + V_k$, the diagonal elements of V_k being zero, we obtain in putting $H_k = -U_k^{-1} V_k$ a generally non-stationary linear iteration process $\xi'_{k+1} = H_k \xi'_k + (E-H_k)A^{-1}\eta'$ which is convergent in such a way that $\xi_{k+1} - \xi_k = 0((\sigma_A + \varepsilon)^k)$ for any positive ε.

11. The one step iteration as defined in the section 1 can be also considered as an iteration of the residual vector. Thus we obtain from (1) and (2) easily

(17) $$\rho'_{k+1} = (E - q_k \cdot \Delta_{N_k})\rho'_k$$

where

(18) $$\Delta_\nu = \begin{pmatrix} 0 & - & - & 0 & a_{1\nu} & 0 & - & - & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & - & - & 0 & a_{\nu-1\,\nu} & 0 & - & - & 0 \\ 0 & - & - & 0 & 1 & 0 & - & - & 0 \\ 0 & - & - & 0 & a_{\nu+1\,\nu} & 0 & - & - & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & - & - & 0 & a_{n,\nu} & 0 & - & - & 0 \end{pmatrix}$$

is the matrix obtained from A in keeping the ν^{th} column and replacing all other elements by zeros. From our theorem II it follows that for an H-matrix A and q_k satisfying the condition (7) the product $\prod_{k=1}^{k}(E - q_k \Delta_{N_k})$ tends with $k \to \infty$ to zero, if the sequence of the leading indices N_k

contains each index 1, ..., n an infinite number of times. By this result we have in particular proven that the Hardy Cross Balancing process for a continuous beam is always convergent if each support is used in balancing an infinite number of times. If in particular the supports are treated in a certain order periodically, the convergence is linear, that is, of the type described at the end of section 10.[6)]

Indeed, in the Hardy Cross process as described by Oldenburger [10] the matrix A is defined by

$$a_{\nu-1\,\nu} = \sigma_\nu \;,\; a_{\nu\nu} = 1,\; a_{\nu+1\,\nu} = \tau_\nu$$
$$a_{\mu\nu} = 0 \;(\mu > \nu+1,\; \mu < \nu-1)$$

where

$$\sigma_i = s_{i-1}(1-t_i),\; \tau_i = r_i t_i$$

$$0 \leq t_i \leq 1,\; 0 \leq r_i < 1,\; 0 \leq s_i < 1.$$

Here we have certainly $\sigma_i + \tau_i < (1-t_1) + t_1 = 1$ and our matrix A is certainly then an H matrix.

11. The proofs for the <u>necessity</u> of our conditions make use of properties of the H matrices given in an earlier paper by the author[11]; however, these properties have to be developed further in different directions.

As to the proofs for the <u>sufficiency</u> of our criteria they use in particular the following lemma:

To any matrix A with the Jacobian constant σ_A and to any positive ε correspond two positive diagonal matrices (i.e. matrices with zeros off the diagonal and positive numbers in the diagonal) P, P_1 such that we have

$$PAP_1 - E = B = (b_{\mu\nu})$$

$$b_{\mu\mu} = 0, \quad \sum_{\nu=1}^{n} |b_{\mu\nu}| \leq \sigma_A + \varepsilon \ , \quad (\mu = 1, \ldots, n).$$

Instead of this lemma also the following one could be used for the same purpose: **Let A be a matrix and ρ the maximum** modulus of the fundamental roots of A, then to any positive ε corresponds a non-singular matrix S such that we have

$$SAS^{-1} = B = (b_{\mu\nu})$$

where

$$\sum_{\nu=1}^{n} |b_{\mu\nu}| \leq \rho + \varepsilon \quad (\mu = 1, \ldots , n).$$

The details of our proofs will be developed in another paper.[7)]

Footnotes

1) As a matter of fact, the rules given by Gauss, Seidel and Southwell for the choice of N are different; but they coincide if the diagonal elements of A are all 1.

2) However, already Nekrasoff [8]had already referred to this iteration /as to the "Seideliteration" and this is the name usually if wrongly used.

3) To this type belongs a class of iterations considered occasionally by H. Geiringer [3a].

4) Cf. [19] for another connection in which these conditions play an important role.

5) This name was proposed to me by Dr. F. Alt.

6) The convergence of the Hardy Cross's Process was proved by Oldenburger [10] in the special case that the balancing process is applied alternatively to all supports with odd numbers and to all supports with even numbers.

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Nekrassof and Mehmke see Mehmke and Nekrassof.

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THE NATIONAL BUREAU OF STANDARDS

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